Exact Green's Function for 2D Dirac Oscillator in Constant Magnetic Field

Salah Haouat and Lyazid Chetouani

Département de Physique, Faculté des Sciences Exactes, Université Mentouri, Route Ain El-Bey, Constantine 25000, Algérie

Reprint requests to L. C.; E-mail: chetoua1@caramail.com

Z. Naturforsch. **62a,** 34 – 40 (2007); received October 16, 2006

The propagator of two-dimensional Dirac oscillator in the presence of a constant magnetic field is presented by means of path integrals, where the spin degree-of-freedom is described by odd Grassmannian variables and the gauge invariant part of the effective action has the form of the standard pseudoclassical action given by Berezin and Marinov. Then the path integration is carried out and the problem is solved exactly. The energy spectrum of the electron and the wave functions are extracted. – PACS numbers: 03.65.Ca, 03.65.Db, 03.65.Pm, 03.65.Ge.

Key words: Path Integral; Dirac Oscillator; Exact Solutions.

1. Introduction

For a long time, the harmonic oscillator has been considered as one of the most useful systems in quantum physics. So there are many relativistic generalizations such as the Dirac oscillator (DO) introduced by Itô et al. [1] and developed by Moshinsky and Szczepaniak [2]. By adding the vector potential $(-\mathrm{i}m\omega\beta\vec{r})$, which is linear in coordinate and carries a matrix β, to Dirac equation, the authors have found a relativistic model, where the nonrelativistic limit reproduces the usual harmonic oscillator.

This model was, during the last 15 years, the subject of many papers and has attracted the attention of many authors. Villalba studied the pure two-dimensional Dirac oscillator (2DDO) [3]. Later, Villalba and Maggiolo computed the energy spectrum of the 2DDO in the presence of a magnetic field [4]. Also, the problem of the 2DDO is solved in the presence of Aharonov-Bohm potential [5]. The Dirac oscillator also has some applications, especially after its second time introduction by Moshinsky and Szczepaniak. For example, Gashimzade and Babaev used this model to express Kane-type semiconductor quantum dots [6]. However, in spite of the 2DDO has been much discussed, there is no well established path integral treatment.

The purpose of this work is to set up a path integration for the problem of the 2DDO in the presence of a constant magnetic field. In Section 2 we give a path integral formulation for the problem in question and we describe the spin degree-of-freedom by the odd Grassmannian variables. In Section 3, after integrating over odd trajectories, we calculate, exactly, Green's function in Cartesian coordinates. Finally, we pass to the polar coordinates in order to extract the related spectrum and wave functions.

2. Path Integral Formulation of 2DDO

The most useful path integral formulation for a relativistic spinning particle interacting with an external field is already proposed by Fradkin and Gitman [7]. They have presented the relative propagator according to the Feynman standard form

$$\int D(\text{path}) \exp \mathrm{i}S(\text{path}), \tag{1}$$

where S is a supersymmetric action, which describes at the same time the external motion and internal one related to the spin of the particle. However, for the problem of the 2DDO, that is governed by the Hamiltonian

$$\mathcal{H}_{\mathrm{DO}} = \vec{\alpha}(\vec{P} - \mathrm{i}m\omega\beta\,\vec{r}) + \beta m, \tag{2}$$

it is convenient to rederive a path integral representation.

Starting from the wave equation that can be written in the form

$$\left[\gamma^{\mu}P_{\mu} + \mathrm{i}m\omega\,\gamma^{0}\vec{\gamma}.\,\vec{r} - m\right]\psi(t,\vec{r}) = 0, \tag{3}$$

where $P_\mu = \mathrm{i}\partial_\mu$ and the γ-matrices are given, in (2+1)-dimension, in terms of Pauli matrices

$$\gamma^0 = \sigma_z, \quad \gamma^1 = \mathrm{i}\sigma_x, \quad \gamma^2 = \mathrm{i}\sigma_y, \tag{4}$$

we define the propagator of the 2DDO in the presence of a constant magnetic field as a causal Green's function $S^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a)$ solution of the equation

$$\begin{aligned}\Big[\gamma^0 P_0 - \vec{\gamma}\left(\vec{P} - e\vec{A}\right) + \mathrm{i}m\omega\gamma^0\vec{\gamma}.\vec{r} - m\Big] S^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) \\ = -\delta(t_b - t_a)\ \delta^2(\vec{r}_b - \vec{r}_a),\end{aligned} \tag{5}$$

where the magnetic field is described by the vector potential

$$\vec{A} = \frac{Br}{2}\tilde{\mathbf{u}}_\vartheta, \tag{6}$$

that has the two components

$$A_x = -\frac{B}{2}y, \quad A_y = \frac{B}{2}x. \tag{7}$$

Then, we present $S^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a)$ as a matrix element of an operator $\mathbb{S}^{\mathrm{c}}$

$$S^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \langle t_b, \vec{r}_b | \mathbb{S}^{\mathrm{c}} | t_a, \vec{r}_a \rangle, \tag{8}$$

where

$$\mathbb{S}^{\mathrm{c}} = -\frac{1}{K_-} = -K_+\frac{1}{K_- K_+}, \tag{9}$$

and the operators K_+ and K_- are given by

$$\begin{aligned}K_\pm = \gamma^0 P_0 - \gamma^1\left(P_1 + \frac{eB}{2}y\right) - \gamma^2\left(P_2 - \frac{eB}{2}x\right) \\ + \mathrm{i}m\omega(\gamma^0\gamma^1 x + \gamma^0\gamma^2 y) \pm m.\end{aligned} \tag{10}$$

Using properties of Pauli matrices, we get, after some calculations,

$$\begin{aligned}K_- K_+ = P^2 - m^2 - m^2\bar{\omega}^2(x^2 + y^2) \\ + 2m\bar{\omega}(P_x y - P_y x) + 2\mathrm{i}m\bar{\omega}\,\gamma^1\gamma^2,\end{aligned} \tag{11}$$

where

$$\bar{\omega} = \omega + \frac{eB}{2m}. \tag{12}$$

Introducing now the relation

$$\int \mathrm{d}t\mathrm{d}\vec{r}\,|t, \vec{r}\rangle\,\langle t, \vec{r}| = 1, \tag{13}$$

we express the propagator $S^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a)$ in the so-called global projection [8]

$$S^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = K_+(t_b, \vec{r}_b)\,G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a), \tag{14}$$

where the new Green's function $G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a)$, that we suggest to calculate via path integration, is defined by

$$G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \langle t_b, \vec{r}_b | \frac{-1}{K_- K_+} | t_a, \vec{r}_a \rangle \tag{15}$$

and has to be represented by the Schwinger proper time method as

$$\begin{aligned}G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \mathrm{i}\int_0^{+\infty} \mathrm{d}\lambda\,\langle t_b, \vec{r}_b| \\ \cdot \exp(-\mathrm{i}\mathcal{H}(\lambda))|t_a, \vec{r}_a\rangle,\end{aligned} \tag{16}$$

where the Hamiltonian $\mathcal{H}(\lambda)$ is given by

$$\begin{aligned}\mathcal{H}(\lambda) = \lambda(-P^2 + m^2 + m^2\bar{\omega}^2(x^2 + y^2) \\ - 2m\bar{\omega}(P_x y - P_y x) - 2\mathrm{i}m\bar{\omega}\gamma^1\gamma^2).\end{aligned} \tag{17}$$

The operator $K_+(t_b, \vec{r}_b)$ will eliminate the superfluous states caused by the product $K_- K_+$.

To present $G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a)$ by means of a path integral, we write, in the beginning, $\exp(-\mathrm{i}\mathcal{H}(\lambda)) = [\exp(-\mathrm{i}\mathcal{H}(\lambda)\varepsilon)]^N$, with $\varepsilon = 1/N$, and we insert $(N-1)$ resolutions of identity $\int |x\rangle\,\langle x|\,\mathrm{d}x = 1$ between all the operators $\exp(-\mathrm{i}\varepsilon\mathcal{H}(\lambda))$. Next, we introduce N additional integrations $\int \mathrm{d}\lambda_k \delta(\lambda_k - \lambda_{k-1}) = 1$. We get

$$\begin{aligned}G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \mathrm{i}\lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \int_0^{+\infty} \mathrm{d}\lambda_0 \int \mathrm{d}x_1\mathrm{d}x_2 \ldots \mathrm{d}x_{N-1} \\ \cdot \int \mathrm{d}\lambda_1\mathrm{d}\lambda_2 \ldots \mathrm{d}\lambda_N \prod_{k=1}^{N} \langle x_k | \exp(-\mathrm{i}\varepsilon\mathcal{H}(\lambda_k)) | x_{k-1}\rangle \\ \cdot \delta(\lambda_k - \lambda_{k-1}).\end{aligned} \tag{18}$$

As ε is small, we can write

$$\langle x_k | \exp(-\mathrm{i}\varepsilon\mathcal{H}(\lambda_k)) | x_{k-1}\rangle \approx \langle x_k | 1 - \mathrm{i}\varepsilon\mathcal{H}(\lambda_k) | x_{k-1}\rangle, \tag{19}$$

and since the operator $\mathcal{H}(\lambda)$ has a symmetric form with respect to operators X and P, the matrix element (19) will be expressed in terms of the Weyl symbols in the middle point $\tilde{x}_k = (x_k + x_{k-1})/2$. In effect, using identities $\int |p_k\rangle \langle p_k| \mathrm{d}p_k = 1$ and taking into account that

$$\langle x_k \mid p_{k'}\rangle = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{\mathrm{i}p_{k'}x_k}, \tag{20}$$

the matrix element in (19) can be written in the form

$$\int \frac{\mathrm{d}p_k}{(2\pi)^3} \exp\left\{\mathrm{i}\left[p_k \frac{x_k - x_{k-1}}{\varepsilon} - \mathcal{H}(\lambda_k, \tilde{x}_k, p_k)\right]\varepsilon\right\}. \tag{21}$$

The multipliers in (18) are noncommutative due to the γ-matrix structure, so that we attribute formally the index k to γ-matrices, and we introduce the $\mathbb{T}$-product which acts on γ-matrices. Then, using the integral representation for the δ-functions

$$\delta(\lambda_k - \lambda_{k-1}) = \frac{\mathrm{i}}{2\pi} \int \mathrm{e}^{\mathrm{i}\pi_k(\lambda_k - \lambda_{k-1})} \mathrm{d}\pi_k, \tag{22}$$

it is possible to gather all the multipliers, entering in (18), in one exponent and the Green's function G^{c} can be expressed as follows:

$$G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \mathbb{T}\int_0^\infty \mathrm{d}\lambda_0 \int Dx \int Dp \int D\lambda \int D\pi \, \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\lambda\left(p^2 - m^2 - m^2\bar{\omega}^2\left(x^2 + y^2\right)\right)\right.\right.$$
$$\left.\left. + \lambda 2m\bar{\omega}(p_x y - p_y x) + \lambda 2m\bar{\omega}\,\gamma^1\gamma^2 + p\dot{x} + \pi\dot{\lambda}\right]\right\}. \tag{23}$$

To insert the γ-matrices by means of paths we introduce in the beginning an odd sources ρ^μ (Grassmannian variable):

$$G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \int_0^\infty \mathrm{d}\lambda_0 \int Dx \int Dp \int D\lambda \int D\pi \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\lambda(p^2 - m^2 - m^2\bar{\omega}^2(x^2 + y^2))\right.\right.$$
$$\left.\left. + \lambda 2m\bar{\omega}(p_x y - p_y x) + p\dot{x} + \pi\dot{\lambda} + \lambda 2m\bar{\omega}\,\frac{\delta_\ell}{\delta\rho^1}\frac{\delta_\ell}{\delta\rho^2}\right]\right\} \mathbb{T}\exp\int_0^1 \rho(\tau)\gamma \mathrm{d}\tau\Big|_{\rho=0}, \tag{24}$$

and we present the quantity $\mathbb{T}\exp\int_0^1 \rho(\tau)\gamma \mathrm{d}\tau$ via a path integral over Grassmannian odd trajectories [7, 8]:

$$\mathbb{T}\exp\int_0^1 \rho(\tau)\gamma \mathrm{d}\tau = \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial\theta^\mu}\right)\int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \exp\left\{\int_0^1 \mathrm{d}\tau\left[\psi_\mu\dot{\psi}^\mu - 2\mathrm{i}\,\rho_\mu\psi^\mu\right] + \psi_\mu(1)\psi^\mu(0)\right\}, \tag{25}$$

where the measure $\mathcal{D}\psi$ is given by

$$\mathcal{D}\psi = D\psi\left[\int_{\psi(0)+\psi(1)=0} D\psi \exp\left\{\int_0^1 \psi_\mu\dot{\psi}^\mu \mathrm{d}\tau\right\}\right]^{-1}, \tag{26}$$

and θ^μ and ψ^μ are odd variables, anticommuting with γ-matrices. Finally, Green's function G^{c} is presented in the Hamiltonian path integral representation

$$G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty d\lambda_0 \int Dx \int Dp \int D\lambda \int D\pi \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\lambda p^2 - \lambda m^2\right.\right.$$
$$\left.\left. - \lambda m^2\bar{\omega}^2(x^2 + y^2) + 2\lambda m\bar{\omega}(p_x y - p_y x) + p\dot{x} + \pi\dot{\lambda} - \mathrm{i}\psi_\mu\dot{\psi}^\mu + \mathrm{i}\lambda 8m\bar{\omega}\,\psi^1\psi^2\right] + \psi_\mu(1)\psi^\mu(0)\right\}\Big|_{\theta=0}. \tag{27}$$

We notice that, integrating over momenta and separating the gauge-fixing term $\pi\dot{\lambda}$ and the boundary term $\psi_\mu(1)\,\psi^\mu(0)$, we obtain the supergauge invariant action [9, 10]

$$\mathcal{A} = \int_0^1 \left[-\frac{\dot{x}^2}{4\lambda} + m\bar{\omega}(y\dot{x} - x\dot{y}) - \mathrm{i}\psi_\mu\dot{\psi}^\mu + \mathrm{i}\lambda 8m\bar{\omega}\,\psi^1\psi^2\right]\mathrm{d}\tau, \tag{28}$$

where the supersymmetric transformation is given by

$$\delta x^\mu = \mathrm{i}\psi^\alpha\varepsilon\,,\ \delta\psi^\alpha = \frac{1}{4\lambda}\dot{x}^\alpha\varepsilon. \tag{29}$$

In the next section, we give an exact calculation of Green's function $G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a)$.

3. The Green's Function

Having shown how to formulate the problem of the 2DDO plus a magnetic field in the framework of Feynman path integrals, let us now go to the calculation of Green's function G^{c}. First, we integrate over π, λ, x_0 and p_0. We obtain

$$G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}E}{2\pi} \mathrm{e}^{-\mathrm{i}E(t_b - t_a)} \times \mathcal{F}(\lambda)\, \mathcal{G}_E(x_b, y_b; x_a, y_a; \lambda), \tag{30}$$

where the function $\mathcal{G}_E$ is given only in terms of bosonic trajectories

$$\begin{aligned} \mathcal{G}_E(x_b, y_b; x_a, y_a; \lambda) = \int DxDy \int Dp_x Dp_y \exp\Big\{ \mathrm{i}\int_0^1 \mathrm{d}\tau \big[\lambda(E^2 - m^2) - \lambda p_x^2 - \lambda p_y^2 - \lambda m^2 \bar{\omega}^2 (x^2 + y^2) \\ + 2\lambda m\bar{\omega}(p_x y - p_y x) + p_x \dot{x} + p_y \dot{y} \big] \Big\}, \end{aligned} \tag{31}$$

and the factor $\mathcal{F}(\lambda)$ is given by

$$\mathcal{F}(\lambda) = \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial \theta^\mu} \right) \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \exp\left\{ \int_0^1 \left[\psi_\mu \dot{\psi}^\mu - 4\lambda F_{\mu\nu} \psi_\mu \psi^\nu \right] \mathrm{d}\tau + \psi_\mu(1)\psi^\mu(0) \right\} \Bigg|_{\theta=0}. \tag{32}$$

Here, the tensor F is defined by $F_{12} = -F_{21} = m\bar{\omega}$ (all other elements are 0) and has to be understood as a matrix with lines marked by the first contravariant indices and with columns marked by the second covariant indices.

In order to calculate $\mathcal{F}(\lambda)$ we change, in the first stage, the integration variables from ψ to ξ, where

$$\psi = \frac{1}{2}\xi + \frac{\theta}{2}, \tag{33}$$

and the new variables ξ obey the following boundary conditions:

$$\xi(0) + \xi(1) = 0. \tag{34}$$

The factor $\mathcal{F}(\lambda)$ will be then given through the Grassmann-Gaussian integral

$$\mathcal{F}(\lambda) = \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial \theta^\mu} \right) \exp(-\lambda F_{\mu\nu}\theta^\mu\theta^\nu) \int \mathcal{D}\xi \exp\left\{ \int_0^1 \left[\frac{1}{4}\xi_\mu \dot{\xi}^\mu - \lambda F_{\mu\nu}\xi^\mu\xi^\nu - 2\lambda F_{\mu\nu}\theta^\mu\xi^\nu \right] \mathrm{d}\tau \right\} \Bigg|_{\theta=0}, \tag{35}$$

that can be computed to be

$$\begin{aligned} \mathcal{F}(\lambda) = \det{}^{\frac{1}{2}}(\cosh 2\lambda F) \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial \theta^\mu} \right) \\ \cdot \left[1 - \lambda B_{\mu\nu}\theta^\mu\theta^\nu \right] \Big|_{\theta=0}, \end{aligned} \tag{36}$$

where the tensor B, that is understood as a matrix, is given by (see [11])

$$B = \frac{1}{2\lambda} \tanh 2\lambda F. \tag{37}$$

From the definition of the tensor $F_{\mu\nu}$, it is easy to show that

$$\cosh 2\lambda F = 1 + \left(\frac{F}{m\bar{\omega}} \right)^2 (1 - \cos 2\lambda m\bar{\omega}) \tag{38}$$

and

$$B = \frac{F}{2\lambda m\bar{\omega}} \tan 2\lambda m\bar{\omega}. \tag{39}$$

We then get

$$\begin{aligned} \mathcal{F}(\lambda) &= \cos(2m\omega\lambda) + \mathrm{i}\gamma^0 \sin(2m\omega\lambda) \\ &= \sum_{s=\pm 1} \frac{1 + s\gamma^0}{2} \exp(\mathrm{i}s 2m\omega\lambda). \end{aligned} \tag{40}$$

Thus, Green's function $G^{c}(t_b,\vec{r}_b,t_a,\vec{r}_a)$ can be expressed only through a bosonic path integral over the space coordinate and their corresponding momenta:

$$G^{c}(t_b,\vec{r}_b,t_a,\vec{r}_a)=\int\frac{dE}{2\pi}e^{-iE(t_b-t_a)}\sum_{s=\pm1}\frac{1+s\gamma^0}{2}\int_0^\infty d\lambda\int DxDy\int Dp_xDp_y\exp\Big\{i\int_0^1 d\tau\Big[\lambda(E^2-m^2)\\ -\lambda p_y^2-\lambda p_x^2-\lambda m^2\bar{\omega}^2(x^2+y^2)+2\lambda m\bar{\omega}(p_xy-p_yx)+p_x\dot{x}+p_y\dot{y}+\lambda 2sm\omega\Big]\Big\}. \tag{41}$$

Having succeeded to do integration over Grassmannian variables, let us now integrate over even (bosonic) trajectories. In the first stage, we integrate over momenta. The equation (31) becomes

$$\mathcal{G}_E(x_b,y_b;x_a,y_a;\lambda)=\exp i\lambda\left(E^2-m^2\right)\int DxDy\exp\left\{i\int_0^1 d\tau\left[\frac{\dot{x}^2}{4\lambda}+\frac{\dot{y}^2}{4\lambda}+m\bar{\omega}(y\dot{x}-x\dot{y})\right]\right\}. \tag{42}$$

The last path integral has a quadratic action and, consequently, is integrable. The result is

$$\mathcal{G}_E(x_b,y_b;x_a,y_a;\lambda)=\exp i\lambda\left(E^2-m^2\right)\frac{m\bar{\omega}}{2\pi\sin(2m\bar{\omega}\lambda)}\exp\left[-im\bar{\omega}(x_ay_b-y_ax_b)\right]\\ \cdot\exp\left\{i\frac{m\bar{\omega}}{2\tan(2m\bar{\omega}\lambda)}\left((x_b-x_a)^2+(y_b-y_a)^2\right)\right\}. \tag{43}$$

4. Energy Spectrum and Wave Functions

In order to obtain the wave functions and their corresponding energies, we must express Green's function $G^{c}(t_b,\vec{r}_b,t_a,\vec{r}_a)$ in polar coordinates ($x=r\cos\vartheta$, $y=r\sin\vartheta$). This passage from Cartesian coordinates to polar ones may be done with the help of the formula

$$\exp\left(-i\frac{m\bar{\omega}r_br_a}{2\tan(2m\bar{\omega}\lambda)}\cos(\Delta\vartheta)\right)=\sum_{k=-\infty}^{+\infty}I_{|k|}\left(\frac{m\bar{\omega}r_br_a}{i\sin(2m\bar{\omega}\lambda)}\right)e^{ik\Delta\vartheta}. \tag{44}$$

We get

$$G^{c}(t_b,\vec{r}_b,t_a,\vec{r}_a)=\sum_{s=\pm1}\frac{1+s\gamma^0}{2}\int\frac{dE}{2\pi}\sum_k\exp i[E(t_b-t_a)+k(\vartheta_b-\vartheta_a)]\\ \cdot\int_0^\infty d\lambda\frac{e^{i\lambda(E^2-m^2+2m\bar{\omega}k+2sm\omega)}}{\sin(2m\bar{\omega}\lambda)}I_{|k|}\left(\frac{m\bar{\omega}r_br_a}{i\sin(2m\bar{\omega}\lambda)}\right)\exp\left[i\frac{m\bar{\omega}}{2}(r_b^2+r_a^2)\cot(2m\bar{\omega}\lambda)\right]. \tag{45}$$

Using now the Hille-Hardy formula [12]

$$\frac{t^{-\alpha/2}}{1-t}\exp\left[-\frac{1}{2}(x+y)\frac{1+t}{1-t}\right]I_\alpha\left(\frac{2\sqrt{xyt}}{1-t}\right)=\sum_{n=0}^{\infty}\frac{t^n n!e^{-\frac{1}{2}(x+y)}}{\Gamma(n+\alpha+1)}(xy)^{\alpha/2}L_n^\alpha(x)L_n^\alpha(y) \tag{46}$$

and taking $t=e^{-i4\lambda m\bar{\omega}}$, $x=m\bar{\omega}r_a^2$, $y=m\bar{\omega}r_b^2$ and $\alpha=|k|$, we obtain a spectral decomposition of Green's function $G^{c}(t_b,\vec{r}_b,t_a,\vec{r}_a)$:

$$G^{c}(t_b,\vec{r}_b,t_a,\vec{r}_a)=\sum_s\frac{1+s\gamma^0}{2}\sum_k\sum_{n=0}^{\infty}\int_0^\infty d\lambda\int\frac{dE}{2\pi}\exp i\lambda[E^2-\mathcal{E}^2]\exp i[k(\vartheta_b-\vartheta_a)-E(t_b-t_a)]\\ \cdot\frac{n!}{\Gamma(n+|k|+1)}e^{-\frac{m\bar{\omega}}{2}(r_b^2+r_a^2)}(m\bar{\omega}r_ar_b)^{|k|}L_n^{|k|}(m\bar{\omega}r_a^2)L_n^{|k|}(m\bar{\omega}r_b^2), \tag{47}$$

where $\mathcal{E} \equiv \mathcal{E}_{n,k,s}$ is given by

$$\mathcal{E}_{n,k,s} = \sqrt{m^2 + 2m\bar{\omega}\,[2n + 1 - s + |k| - k]}. \quad (48)$$

The quantity $\frac{1}{2}\left(1 + s\gamma^0\right)$ can be written in a product form of a spinor u and it's conjugate $\bar{u}$:

$$\frac{1 + s\gamma^0}{2} = u_s \bar{u}_s, \quad (49)$$

where

$$u_{s=-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_{s=+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (50)$$

Then, by integration over λ we get the spectral decomposition of Green's function:

$$G^{\mathrm{c}}\left(t_b, \vec{r}_b, t_a, \vec{r}_a\right) = \sum_s \sum_k \sum_{n=0}^{\infty} \int \frac{\mathrm{d}E}{2\pi} \frac{\varphi_{n,l,s}\left(t_b, \vec{r}_b\right) \bar{\varphi}_{n,l,s}\left(t_a, \vec{r}_a\right)}{E^2 - \mathcal{E}^2 + \mathrm{i}\varepsilon}, \quad (51)$$

where

$$\varphi_{n,k,s}\left(t, \vec{r}\right) = u_s \mathrm{e}^{-\mathrm{i}Et} \mathrm{e}^{\mathrm{i}k\vartheta} \phi_{n,k}(r) \quad (52)$$

and the functions $\phi_{n,k}(r)$ are the radial part of wave functions relative to the nonrelativistic two-dimensional radial harmonic oscillator

$$\phi_{n,k}(r) = C_{n,k}\, \mathrm{e}^{-\frac{m\bar{\omega}}{2} r^2} \left(\sqrt{m\bar{\omega}} r\right)^{|k|} L_n^{|k|}\left(m\bar{\omega} r_a^2\right) \quad (53)$$

with

$$C_{n,k} = \sqrt{\frac{n!}{\Gamma\left(n + |k| + 1\right)}}. \quad (54)$$

Integrating over E, we obtain

$$G^{\mathrm{c}}(t_b, \vec{r}_b, t_a, \vec{r}_a) = \sum_{\varepsilon=\pm 1} \sum_{s=\pm 1} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \Theta[\varepsilon(t_b - t_a)]\, \psi^{\varepsilon}_{n,k,s}(t_b, \vec{r}_b) \bar{\psi}^{\varepsilon}_{n,k,s}(t_a, \vec{r}_a), \quad (55)$$

where $\Theta\left(x\right)$ is the Heaviside step function and

$$\psi^{\varepsilon}_{n,k,s}\left(t, \vec{r}\right) = u_s \mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{E}t} \mathrm{e}^{\mathrm{i}k\vartheta} \phi_{n,k}(r). \quad (56)$$

Now we must determine the Dirac oscillator states by acting the operator K_+ on the functions $\psi^{\varepsilon}_{n,k,s}\left(t, \vec{r}\right)$:

$$\Psi^{\varepsilon}_{n,k,s} = \mathcal{N} K_+ \varphi^{\varepsilon}_{n,k,s}\left(t, \vec{r}\right), \quad (57)$$

where $\mathcal{N}$ is a normalization constant.

Writing K_+ in the form

$$\begin{aligned} K_+ &= \gamma^0 \left(\mathrm{i}\frac{\partial}{\partial t}\right) + m \\ &+ \frac{\gamma^1 + \mathrm{i}\gamma^2}{2} \mathrm{e}^{-\mathrm{i}\vartheta} \left(\mathrm{i}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \vartheta} + \mathrm{i}m\bar{\omega}r\right) \\ &+ \frac{\gamma^1 - \mathrm{i}\gamma^2}{2} \mathrm{e}^{\mathrm{i}\vartheta} \left(\mathrm{i}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial \vartheta} - \mathrm{i}m\bar{\omega}r\right), \end{aligned} \quad (58)$$

the spinors $\psi^{\varepsilon}_{s=+1}$ and $\psi^{\varepsilon}_{s=-1}$ will be then written as

$$\Psi^{\varepsilon}_{n,k,s=+1} = \mathrm{i}\mathcal{N}\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{E}t} \begin{pmatrix} (\varepsilon\mathcal{E} + m)\mathrm{e}^{\mathrm{i}k\vartheta} \\ -\mathrm{e}^{\mathrm{i}(k-1)\vartheta}\left(\frac{\partial}{\partial r} + \frac{k}{r} + m\bar{\omega}r\right) \end{pmatrix} \phi_{n,k}(r), \quad (59)$$

$$\Psi^{\varepsilon}_{n,k,s=-1} = \mathrm{i}\mathcal{N}\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{E}t} \begin{pmatrix} -\mathrm{e}^{\mathrm{i}(k+1)\vartheta}\left(\frac{\partial}{\partial r} - \frac{k}{r} - m\bar{\omega}r\right) \\ \mathrm{e}^{\mathrm{i}k\vartheta}(-\varepsilon\mathcal{E} + m) \end{pmatrix} \phi_{n,k}(r). \quad (60)$$

Using the differential formulas of the functions $\phi_{n,k}(r)$ (see Appendix), we obtain, finally, our solutions that can be classified as follows:

(i) If k is a positive integer, we have

$$\Psi^{\varepsilon}_{n,k,s=+1} = \mathrm{i}\mathcal{N}\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{E}t} \begin{pmatrix} (\varepsilon\mathcal{E} + m)\mathrm{e}^{\mathrm{i}k\vartheta}\phi_{n,k}(r) \\ -2\,\mathrm{e}^{\mathrm{i}(k-1)\vartheta}\sqrt{m\bar{\omega}(n+k)}\phi_{n,k-1}(r) \end{pmatrix}, \quad (61)$$

$$\Psi^{\varepsilon}_{n,k,s=-1} = \mathrm{i}\mathcal{N}\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{E}t} \begin{pmatrix} 2\sqrt{m\bar{\omega}(n+k+1)}\mathrm{e}^{\mathrm{i}(k+1)\vartheta}\phi_{n,k+1}(r) \\ (-\varepsilon\mathcal{E} + m)\mathrm{e}^{\mathrm{i}k\vartheta}\phi_{n,k}(r) \end{pmatrix}. \quad (62)$$

(ii) If k has a negative value, the two spinors $\psi^{\varepsilon}_{s=+1}$, $\psi^{\varepsilon}_{s=-1}$ will be given by

$$\Psi^{\varepsilon}_{n,k,s=+1} = \mathrm{i}\mathcal{N}\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{E}t} \begin{pmatrix} (\varepsilon\mathcal{E} + m)\mathrm{e}^{\mathrm{i}k\vartheta}\phi_{n,k}(r) \\ 2\sqrt{m\bar{\omega}n}\mathrm{e}^{\mathrm{i}(k-1)\vartheta}\phi_{n-1,k+1}(r) \end{pmatrix}, \quad (63)$$

$$\Psi^{\varepsilon}_{n,k,s=-1} = \mathrm{i}\mathcal{N}\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{E}t} \begin{pmatrix} -2\sqrt{m\bar{\omega}(n+1)}\mathrm{e}^{\mathrm{i}(k+1)\vartheta}\phi_{n+1,k-1}(r) \\ (-\varepsilon\mathcal{E} + m)\mathrm{e}^{\mathrm{i}k\vartheta}\phi_{n,k}(r) \end{pmatrix}. \quad (64)$$

Let us remark that our spinors are expressed in a fixed Cartesian representation and the passage to the diagonal (rotating) representation, that is used in [3] and [4], can be done by means of the following transformation:

$$\Psi_{\text{diag}} = \sqrt{r}S(\vartheta)^{-1}\Psi^{\varepsilon}_{n,k,s}, \tag{65}$$

where

$$S(\vartheta) = \exp\left(-\frac{\vartheta}{2}\gamma^1\gamma^2\right). \tag{66}$$

5. Conclusion

We have solved the problem of the 2D Dirac oscillator in the presence of a constant magnetic field by using the Feynman-Beresin path integrals. In the first stage we have given a pseudoclassical action path integral, where we have described the spin degrees-of-freedom by fermionic variables (Grassmannian variables). Then, the integration over odd trajectories has been done easily and the exact Green's function is calculated in Cartesian coordinates. The passage to the polar coordinates permits to extract the energy spectrum of the electron and the corresponding wave functions.

It is obvious that this path integral treatment has two advantages: the first one is to determine the relative Green's function which leads to the good comprehension of the quantum behavior of this system, and the second one is that the spinors that we extract from the spectral decomposition of Green's function have simple forms which make them suitable for variational calculations of physical quantities.

In conclusion, the list of solvable relativistic problems in (2+1)-dimension by using path integrals has been extended to the Dirac oscillator in the presence of a magnetic field. Let us remark that the problem of the three-dimensional DO is under consideration.

Appendix: Important Differential Relations

With the use of [12]

$$x\frac{\partial}{\partial x}L_n^k(x) = nL_n^k(x) - (n+k)L_{n-1}^k(x) \tag{67}$$

and

$$L_n^k(x) = L_n^{k+1}(x) - L_{n-1}^{k+1}(x) \tag{68}$$

we get

$$\begin{aligned}\left(\frac{\partial}{\partial r} + \frac{|k|}{r} + m\bar{\omega}r\right)\phi_{n,k}(r) \\ = 2\sqrt{m\bar{\omega}(n+|k|)}\phi_{n,k-1}(r),\end{aligned} \tag{69}$$

$$\begin{aligned}\left(\frac{\partial}{\partial r} + \frac{|k|}{r} - m\bar{\omega}r\right)\phi_{n,k}(r) \\ = 2\sqrt{m\bar{\omega}(n+1)}\phi_{n+1,k-1}(r),\end{aligned} \tag{70}$$

$$\begin{aligned}\left(\frac{\partial}{\partial r} - \frac{|k|}{r} + m\bar{\omega}r\right)\phi_{n,k}(r) \\ = -2\sqrt{m\bar{\omega}n}\phi_{n-1,k+1}(r),\end{aligned} \tag{71}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial r} - \frac{|k|}{r} - m\bar{\omega}r\right)\phi_{n,k}(r) \\ = -2\sqrt{m\bar{\omega}(n+|k|+1)}\phi_{n,k+1}(r).\end{aligned} \tag{72}$$

[1] D. Itô, K. Mori, and E. Carreri, Nuovo Cim. A **51**, 1119 (1967).
[2] M. Moshinsky and A. Szczepaniak, J. Phys. A **22**, 817 (1989).
[3] V. M. Villalba, Phys. Rev. A **49**, 586 (1994).
[4] V. M. Villalba and A. R. Maggiolo, Eur. Phys. J. B **22**, 31 (2001).
[5] N. Ferkous and A. Bounames, Phys. Lett. A **325**, 21 (2001).
[6] F. M. Gashimzade and A. M. Babaev, Phys. Solid State **44**, 162 (2002).
[7] E. S. Fradkin and D. M. Gitman, Phys. Rev. D **44**, 3220 (1991).
[8] C. Alexandrou, R. Rosenfelder, and A. W. Schreiber, Phys. Rev. A **59**, 3 (1998).
[9] F. A. Berezin and M. S. Marinov, JETP Lett. **21**, 320 (1975); Ann. Phys. **104**, 336 (1977).
[10] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, and P. Howe, Phys. Lett. B **64**, 435 (1976); L. Brink, P. Di Vecchia, and P. Howe, Nucl. Phys. B **118**, 76 (1977).
[11] D. M. Gitman and S. I. Zlatev, Phys. Rev. D **55**, 7701 (1997); D. M. Gitman, S. I. Zlatev, and W. D. Cruz, Brazilian J. Phys. **26**, 419 (1996).
[12] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York 1979.